A global optimality result with application to orbital transfer

Bernard Bonnard, Jean-Baptiste Caillau

A global optimality result with application to orbital transfer

Bernard Bonnard [a], Jean-Baptiste Caillau [b]

[a]*Institut de mathématiques de Bourgogne (UMR CNRS 5584), 9 avenue Savary, F-21078 Dijon*
[b]*ENSEEIHT-IRIT (UMR CNRS 5505), 2 rue Camichel, F-31071 Toulouse*

Received *****; accepted after revision +++++

Presented by

Abstract

The objective of this note is to present a global optimality result on Riemannian metrics $ds^2 = dr^2 + (r^2/c^2)(G(\varphi)d\theta^2 + d\varphi^2)$. This result can be applied to the averaged energy minimization coplanar orbit transfer problem. *To cite this article: B. Bonnard, J.-B. Caillau, C. R. Acad. Sci. Paris, Ser. I xxx (200x).*

Résumé

Un résultat d'optimalité globale et son application en transfert orbital. Cette note présente un résultat d'optimalité globale pour les métriques riemanniennes de la forme $ds^2 = dr^2 + (r^2/c^2)(G(\varphi)d\theta^2 + d\varphi^2)$. Ce résultat s'applique au problème de la minimisation de l'énergie en transfert orbital. *Pour citer cet article : B. Bonnard, J.-B. Caillau, C. R. Acad. Sci. Paris, Ser. I xxx (200x).*

Version française abrégée

Nous présentons dans cette Note un résultat d'optimalité globale pour les métriques riemanniennes de la forme $ds^2 = dr^2 + (r^2/c^2)(G(\varphi)d\theta^2 + d\varphi^2)$. Ce résultat a des applications en mécanique spatiale, dans le cadre du transfert orbital entre orbites kepleriennes. On sait en effet dans ce cas que le moyenné du système hamiltonien décrivant les transferts à énergie minimale [7] est riemannien [2], et qu'il existe des coordonnées orthogonales (voir (1)). Par analogie avec les coordonnées sphériques pour la métrique plate en dimension trois, nous obtenons alors la forme normale annoncée qui joue le rôle de compactification sur $\mathbf{S}^2$ d'une partie de la métrique de départ.

Nous vérifions l'intégrabilité au sens de Liouville des métriques (2), et donnons un algorithme d'intégration par projection du système sur la sous-variété riemannienne de dimension deux $\{r = c\} \simeq \mathbf{S}^2$, en utilisant le fait que la coordonnée r^2 est polynomiale en temps. La métrique résultant de la projection est une métrique de Clairaut-Liouville qui généralise les métriques obtenues en restreignant la métrique plate à une surface de révolution [1].

Email addresses: `Bernard.Bonnard@u-bourgogne.fr` (Bernard Bonnard), `caillau@n7.fr` (Jean-Baptiste Caillau).

L'intégrale première linéaire du système associée à la coordonnée cyclique θ vérifie en particulier la relation classique de Clairaut, et nous calculons également la courbure de Gauss dans le cas du transfert orbital, courbure dont on montre qu'elle est maximale à l'équateur.

Après en avoir déduit une première estimation des temps conjugués du système, nous caractérisons la ligne de partage en dimension trois grâce à celle de la métrique de Clairaut-Liouville. Nous montrons ensuite dans le contexte du transfert que cette métrique est conforme à la restriction de la métrique plate à un ellipsoïde oblat de demi-petit axe $1/\sqrt{5}$, ce qui nous permet d'estimer le rayon d'injectivité dans ce cas. Nous donnons alors le résultat final sous la forme d'une condition nécessaire d'optimalité globale pour les métriques riemanniennes de la forme (2) portant sur le rayon d'injectivité de la métrique de Clairaut-Liouville associée, et nous en déduisons finalement l'existence de points de coupure pour la métrique compactifiée du transfert orbital.

1. Introduction

In our previous article [2], we presented preliminary results concerning the averaged energy minimization problem for the coplanar transfer between Keplerian elliptic orbits [7] (see also [3] for similar results in the single-input case). The extremal curves are shown to be integral curves of the averaged Hamiltonian

$$H = \frac{1}{2n^{5/3}}\left[9n^2p_n^2 + \frac{5}{2}(1-e^2)p_e^2 + (5-4e^2)\frac{p_\theta^2}{2e^2}\right],$$

corresponding to the Riemannian metric

$$g = \frac{1}{9n^{1/3}}dn^2 + \frac{2n^{5/3}}{5(1-e^2)}de^2 + \frac{2n^{5/3}}{5-4e^2}e^2d\theta^2 \tag{1}$$

for which the *orbital elements* $x = (n,e,\theta)$ (see [9]) define orthogonal coordinates: $n = a^{-3/2}$ is the *mean movement* (a is the semi-major axis), e is the *eccentricity*, and θ the *argument of perigee*. These coordinates are singular for $e = 0$, that is for circular orbits, and the singularity can be removed by the polar blowing up $e_x = \cos\theta$, $e_y = \sin\theta$.

A crucial result in our analysis is the construction of a normal form. If we set

$$r = \frac{2}{5}n^{5/6} \quad \text{and} \quad \varphi = \arcsin e,$$

the metric is isometric to

$$g = dr^2 + \frac{r^2}{c^2}(G(\varphi)d\theta^2 + d\varphi^2) \tag{2}$$

with $c = \sqrt{2/5}$ and where

$$G(\varphi) = \frac{5\sin^2\varphi}{1+4\cos^2\varphi}.$$

A similar result is valid in the single-input case. This normal form captures the main properties of the orbital transfer. Indeed, this metric decomposes into $g_1 = dr^2 + r^2d\psi^2$ (with $\psi = \varphi/c$) and $g_2 = G(\varphi)d\theta^2 + d\varphi^2$. The polar metric g_1 is isometric to the flat metric and associated to transfer towards circular orbits where the argument of the pericenter can be set to zero. The metric g_2 has to be analyzed to conclude about general transfers. It acts as a compactification on $\mathbf{S}^2$ of the initial metric since θ and φ can be interpretated as angular coordinates.

The normal form also reveals that g is integrable by quadratures and gives an algorithm to integrate the extremal flow. The first step is to compute the evolution of r which is independent of the other variables. Then, we derive the evolution of θ and φ using a reparameterization depending upon r. The computation of the geodesics of $g_2 = G(\varphi)d\theta^2 + d\varphi^2$ is then standard. Besides, such geodesics share properties with geodesics on a surface of revolution where φ is the angle along a meridian, and θ along a parallel (see [5,1]).

In this note whose objective is to provide a refined analysis of the averaged energy minimization problem, we present a global optimality result for metrics of the given normal form, $dr^2+(r^2/c^2)(G(\varphi)d\theta^2+d\varphi^2)$, related to the construction of spheres for metrics $G(\varphi)d\theta^2+d\varphi^2$. In this construction, one needs to make an estimate of the conjugate and cut loci of such metrics so as to compute the injectivity radius. Another key point in our analysis is to put in correspondence the geodesics of our problem with geodesics of the flat metric in spherical coordinates, that is $dr^2+r^2(\sin^2\varphi d\theta^2+d\varphi^2)$.

2. Geometric preliminaries

An important step in our analysis is to integrate the geodesics flow of the metric $ds^2=dr^2+(r^2/c^2)(G(\varphi)d\theta^2+d\varphi^2)$ using the following algorithm. The associated Hamiltonian is decomposed into

$$H=\frac{1}{2}p_r^2+\frac{c^2}{r^2}H_2,\quad H_2=\frac{1}{2}\left(\frac{p_\theta^2}{G(\varphi)}+p_\varphi^2\right),$$

and the Liouville integrability is a consequence of the following lemma.

Lemma 2.1 *The Hamiltonian vector field $\overrightarrow{H}$ admits three independent first integrals in involution, H, H_2 and p_θ.*

To compute a complete parameterization, we rewrite the extremals according to

$$\begin{aligned}
\dot r &= p_r, & \dot p_r &= \frac{2c^2}{r^3}\left(\frac{p_\theta^2}{G(\varphi)}+p_\varphi^2\right),\\
\dot\theta &= \frac{c^2}{r^2}\frac{\partial H_2}{\partial p_\theta}, & \dot p_\theta &= -\frac{c^2}{r^2}\frac{\partial H_2}{\partial\theta},\\
\dot\varphi &= \frac{c^2}{r^2}\frac{\partial H_2}{\partial p_\varphi}, & \dot p_\varphi &= -\frac{c^2}{r^2}\frac{\partial H_2}{\partial\varphi}.
\end{aligned}$$

Considering the equations associated with (r,p_r), the result hereafter is obvious.

Lemma 2.2 *The function r^2 is a degree two polynomial in time, and*

$$r^2(t)=t^2+2r_0p_{r0}t+r_0^2$$

on the level set $\{H=1/2\}$.

An important observation is that the solution depends on r_0 and p_{r0}, not on the function G. The remaining equations are then integrated using the reparameterization $d\tau=c^2dt/r^2$, and correspond to integral curves of $\overrightarrow{H}_2$.

Lemma 2.3 *Setting $p_{r0}=\sin\alpha_0$ (and excluding $p_{r0}=\pm1$),*

$$\tau(t,r_0,p_{r0})=\frac{c^2}{r_0\cos\alpha_0}\left[\arctan\left(\frac{t}{r_0\cos\alpha_0}+\tan\alpha_0\right)-\alpha_0\right].$$

To conclude the computation, we must integrate $\overrightarrow{H}_2$ by quadratures. We introduce the following *ad hoc* geometric concept.

Definition 2.4 *We call Clairaut-Liouville a two-dimensional metric normalizable to*

$$ds^2=G(\varphi)d\theta^2+d\varphi^2.$$

Such metrics were obtained by Darboux [5] when restricting the flat metric to a surface of revolution. In this case, θ is the angle along the equator and φ along meridians. Conversely, in order to interpret so a metric $ds^2=G(\varphi)d\theta^2+d\varphi^2$, we proceed as follows. The surface is parameterized by

$$x=F(\varphi)\cos\theta,\quad y=F(\varphi)\sin\theta,\quad z=h(\varphi),$$

where $F=\sqrt{G}$. One has

$$dx^2+dy^2+dz^2=G(\varphi)d\theta^2+(h'^2(\varphi)+F'^2(\varphi))d\varphi^2,$$

so we must impose $h'^2+F'^2=1$. Hence we get the compatibility condition $|F'|\leq 1$ which is not always fulfilled. In particular, it is clearly not satisfied in the transfer case. Nevertheless, the integrability is a consequence of the classical Clairaut relation on a surface of revolution.

Lemma 2.5 *The linear first integral p_θ verifies*

$$p_\theta = \cos\phi\sqrt{G}$$

where ϕ is the angle with respect to a parallel of a geodesic parameterized by arc length.

Lemma 2.6 *The only local covariant of a Clairaut-Liouville metric $G(\varphi)d\theta^2+d\varphi^2$ is the Gauss curvature, K. For $G(\varphi)=5\sin^2\varphi/(1+4\cos^2\varphi)$,*

$$K=\frac{5(1-8\cos^2\varphi)}{(1+4\cos^2\varphi)^2}$$

and $K\leq 5$. Hence the curvature is maximum on the equator ($\varphi=\pi/2$), and the distance to the first conjugate point is greater than or equal to $\pi/\sqrt{5}$.

3. Global optimality result in orbital transfer

We recall the following standard concepts of Riemannian geometry (see [6,8]) which shall be used in the sequel.

Definition 3.1 *Let (M,g) be a Riemannian manifold. If x_0 belongs to M, we denote by $C(x_0)$ the* conjugate locus *formed by the set of first points conjugate to x_0. The* separating line, *$L(x_0)$, is the set of points where two minimizing extremals departing from x_0 intersect, and the* cut locus, $\mathrm{Cut}(x_0)$, *is the set of points where extremals starting from x_0 cease to be globally optimal. If $i(x_0)$ stands for the distance between x_0 and its cut locus, the* injectivity radius *is*

$$i(X)=\inf_{x_0\in M} i(x_0).$$

Proposition 3.1 *Assume the Riemannian manifold complete. Then, the following properties hold.*

(i) A cut point is either on the separating line or a conjugate point.

(ii) If x_1 is a point that realizes the distance from x_0 to its cut locus, then either x_1 is conjugate to x_0, or there are two minimizing geodesics joining x_0 to x_1 and forming the two halves of a same closed geodesic.

(iii) The distance $i(x_0)$ is the smallest radius for which the sphere is not smooth.

Consider a metric of the form $ds^2=dr^2+(r^2/c^2)(G(\varphi)d\theta^2+d\varphi^2)$. We fix as before the parameterization to arc length by restricting to the level set $\{H=1/2\}$. Let x_1 and x_2 be two extremal curves starting from the same initial point x_0 and intersecting at some positive $\bar{t}$. We get three relations, $r_1(\bar{t})=r_2(\bar{t})$, $\theta_1(\bar{t})=\theta_2(\bar{t})$, $\varphi_1(\bar{t})=\varphi_2(\bar{t})$, and deduce from Lemma 2.2 the following.

Proposition 3.2 *Both extremals x_1 and x_2 share the same p_{r0} and, for each t, $r_1(t)=r_2(t)$.*

If we consider now the integral curves of $\overrightarrow{H}_2$ in the fixed induced level $\{H_2=r_0^2(1-p_{r0}^2)/(2c^2)\}$, and parameterize these curves using $d\tau=c^2dt/r^2$, we deduce the following characterization.

Proposition 3.3 *The following are necessary and sufficient conditions to characterize the separating line of the metric $dr^2+(r^2/c^2)(G(\varphi)d\theta^2+d\varphi^2)$:*

$$\theta_1(\bar{\tau})=\theta_2(\bar{\tau}),\quad \varphi_1(\bar{\tau})=\varphi_2(\bar{\tau}),$$

with the compatibility condition that

$$\bar{\tau}=\frac{c^2}{r_0\cos\alpha_0}\left[\arctan\left(\frac{\bar{t}}{r_0\cos\alpha_0}+\tan\alpha_0\right)-\alpha_0\right]\quad \textit{where}\quad p_{r0}=\sin\alpha_0,$$

for some positive $\bar{t}$.

The objective of this section is to get a global optimality result that we can apply to our orbit transfer problem. To this end, we interpret the extremals of the Clairaut-Liouville as extremals of a global metric on the sphere. This is possible in our case since we have compactified the original metric by setting $e = \sin\varphi$, thus defining global coordinates (θ, φ) on $\mathbf{S}^2$ with the usual singularities at the poles. As previously noticed in Lemma 2.6, the lowest bound for conjugate times is $\pi/\sqrt{5}$, and this bound is the sharpest. Indeed, the equator $\varphi = \pi/2$ is an integral curve of the Hamiltonian H_2 on which the curvature is constant and maximum, equal to 5.

More generally, the conjugate locus can be computed using the algorithms described in [4], and is related to the Gauss curvature. Though the curvature takes negative values, it can be related to the restriction of the flat metric on the sphere, or more precisely on an appropriate ellipsoid. Write indeed the metric as

$$\frac{1}{E_\mu(\varphi)}(\sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2)$$

where $E_\mu(\varphi) = \mu^2 + (1-\mu^2)\cos^2\varphi$ and $\mu = 1/\sqrt{5}$. The metric $\sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2$ is the restriction of the flat metric to the oblate ellipsoid of revolution with unit semi-major axis and semi-minor axis μ, embedded in $\mathbf{R}^3$ according to

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \mu\cos\varphi.$$

Proposition 3.4 *The Clairaut-Liouville metric of the transfer is conformal to the flat metric restricted to the corresponding ellipsoid with $\mu = 1/\sqrt{5}$.*

We now compare properties of both metrics. The curvature in the flat case is

$$K = \frac{\mu^2}{(\mu^2 + (1-\mu^2)\cos^2\varphi)^2}$$

and the maximum $1/\mu^2$ is reached on the equator, $\varphi = \pi/2$. Hence the first conjugate point has length $\pi/\sqrt{5}$, as in the transfer case. The respective lengths of the periodic solutions corresponding to the equator or the meridians are the following : 2π for the flat case versus $2\pi\sqrt{5}$ for the orbit transfer along the equator. As for meridians, the length is given by the elliptic integral

$$\int_0^{2\pi}(\mu^2 + (1-\mu^2)\cos^2\varphi)^{1/2}d\varphi$$

in the flat case, clearly greater than $\pi/\sqrt{5}$, and 2π for the transfer. Those estimates have to be used to evaluate the distance to the cut locus.

Lemma 3.2 *In both cases, on the ellipsoid with $\mu = 1/\sqrt{5}$, the injectivity radius is $\pi/\sqrt{5}$. It corresponds to the length of the first conjugate point along the equator.*

To conclude on global optimality, we use the fact that the cut locus on a two-dimensional Riemannian manifold is obtained as the closure of the separating line, and get the following.

Theorem 3.3 *A necessary global optimality condition for a metric normalized to*

$$ds^2 = r^2 + (r^2/c^2)(G(\varphi)d\theta^2 + d\varphi^2)$$

on $\mathbf{R}_+^* \times \mathbf{S}^2$ is that the injectivity radius be greater than or equal to $c\pi$ on $\{r = c\} \simeq \mathbf{S}^2$, the bound being reached by the flat metric in spherical coordinates.

In orbital transfer, the injectivity radius is $\pi/\sqrt{5} < \pi\sqrt{2/5}$, and there are cut points on the compactified metric. For a given initial condition, we make numerical simulations to evaluate the distance to the conjugate locus on the ellipsoid so as to estimate the injectivity radius. If it is less than $\pi\sqrt{2/5}$, the sphere is not smooth and singularities propagate along the cut locus. Since the compactification augments the initial domain of trajectories with the parabolic boundary ($e = \pm 1$, corresponding to the equator), the final analysis requires to check wether such singularities remain in the domain of elliptic trajectories or not.

References

[1] A. Bolsinov and A. Fomenko, *Integrable geodesic flows on two-dimensional surfaces*, Kluwer, New-York, 2000.

[2] B. Bonnard and J.-B. Caillau, Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust, *Annales de l'Institut Henri Poincaré*, to appear.

[3] B. Bonnard, J.-B. Caillau and R. Dujol, Energy minimization of single input orbit transfer by averaging and continuation, *Bulletin des Sciences Mathématiques*, to appear.

[4] B. Bonnard, J.-B. Caillau and E. Trélat, Second order optimality conditions in the smooth case and applications in optimal control, *Control, Optimisation and Calculus of Variations*, to appear.

[5] G. Darboux, *Leçons sur la théorie générale des surfaces*, Tome III, Gauthiers-Villars, 1914.

[6] M. P. Do Carmo, *Riemannian geometry*, Birkhäuser, 1992.

[7] R. Epenoy and S. Geffroy, Optimal low-thrust transfers with constraints: generalization of averaging techniques, *Acta Astronautica* Vol. 41:3, pp. 133-149, 1997.

[8] W. Klingenberg, *Riemannian geometry*, de Gruyter, 1982.

[9] O. Zarrouati, *Trajectoires spatiales*, CNES–Cepadues, Toulouse, 1987.